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Broer, H.; Hoveijn, I.; Lunter, G.; Vegter, G.

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Appendix

A.1 Classification of term orders

A term order $<$ for Gröbner bases is required to have the following properties:

1. $m \geq 1$, (Noetherian)
2. If $m' \leq m''$ then $mm' \leq mm''$, (Multiplicative)
3. The order $<$ is a total order.

(For the definition of total order, see Sect. 6.3.2.) Such orders are called *admissible orders*. For term orders in formal power series rings, property 1, Noetherianness, is not required. Here we shall use the term *admissible order* to refer to term orders satisfying property 2 and 3. It turns out to be possible to classify all admissible term orders, and to describe them in a uniform way.

In order to formulate the result, we need the following ordering of univariate polynomials. For polynomials $f, g \in \mathbb{R}[Z]$, define $f \geq g$ if and only if $\text{LC}(f - g) \geq 0$, where the ordinary term ordering for univariate polynomials ($1 < Z < Z^2 < \dots$) is supposed. This is equivalent to the perhaps more intuitive definition

$$f \geq g \quad \Leftrightarrow \quad f = g \text{ or } f(x) - g(x) \rightarrow \infty \quad (x \rightarrow \infty).$$

Let $\eta_i \in \mathbb{R}[Z]$ for $i = 1, \dots, n$ be n rationally independent polynomials, then a term order on the variables x_1, \dots, x_n , or equivalently on vectors in \mathbb{N}^n , can be defined as follows:

Definition A.1. (Term order $<_\eta$) For $\alpha, \beta \in \mathbb{N}^n$, let $\alpha <_\eta \beta$ if and only if $\sum_i \alpha_i \eta_i < \sum_i \beta_i \eta_i$.

The result of this section is that this is in fact the most general way of defining term orders:

Proposition A.2. For every admissible term order $<$ on x_1, \dots, x_n there exists a vector of univariate polynomials η such that $<$ coincides with $<_\eta$. Moreover, the polynomials have degree at most $n - 1$.

This result is proved in [Rob85] and [Wei87]. Here we give a different proof using nonstandard analysis (see [Rob88] for a nice introduction), which is both shorter and, in our opinion, more intuitive. The idea is as follows. It is easy to show that

a linear form on vectors in \mathbb{N}^n satisfying some nondegeneracy conditions defines a term order. Conversely, every term order is ‘close’ to such a order defined by a linear form, in the sense that given any term order and a *finite* set of monomials, there exists a linear form such that the corresponding term order coincides with the given order on the finite set. The actual term order is therefore a ‘limit’ of such linear form orders, in the appropriate sense. Taking the limit is technically unpleasant, but this part becomes straightforward using nonstandard analysis.

The first part of the proof is based on the proof of [Stu96, Proposition 1.11], for which we need the following familiar lemma:

Lemma A.3 (Farkas). [Sch86, Ch. 7.3] *Let A be a matrix and b a row vector. Then there exists a row vector $y \geq 0$ with $yA = b$ if and only if $bx \geq 0$ for every column vector x with $Ax \geq 0$.*

Here $x \geq 0$ for a vector x means that every component of x is nonnegative.

Proof of proposition A.2: Let $M \in \mathbb{N}$ be unbounded, and let α_i for $i = 1, \dots, M^n$ be the elements of $[0, M-1]^n \subset \mathbb{N}^n$ ordered in such a way that $i < j \Leftrightarrow \alpha_i < \alpha_j$. Define $\alpha_{ij} := \alpha_j - \alpha_i$, and

$$C_{ij} := \{x \in \mathbb{R}^n \mid \alpha_{ij} \cdot x \geq 0\},$$

$$C := \bigcap_{1 \leq i < j \leq M^n} C_{ij}.$$

(The set C_{ij} contains all vectors x such that $\alpha_i \leq_x \alpha_j$, where x is interpreted as a vector of polynomials of degree 0.) We claim that $C \neq \{0\}$. Suppose on the contrary that C only contains the zero vector. This means that there is no nonzero vector x such that $\alpha_{ij} \cdot x \geq 0$ for all $i < j$. If A is the matrix consisting of the row vectors α_{ij} with $i < j$, this implies that the condition of Farkas’ lemma is trivially satisfied, so there exist nonnegative real numbers y_{ij} such that

$$(A.1) \quad \sum_{1 \leq i < j \leq M^n} y_{ij} \alpha_{ij} = (-1, -1, \dots, -1).$$

Since the α_{ij} are rational, we may suppose that the y_{ij} are too. Then, by clearing denominators, (A.1) can be written as

$$(A.2) \quad \sum_{1 \leq i < j \leq M^n} y'_{ij} \alpha_{ij} = -b,$$

where the y'_{ij} are nonnegative integers, and b is some nonnegative integer vector.

Note that the multiplicative property and total orderedness together imply the more general

$$m_1 < m_2 \text{ and } m_3 \leq m_4 \quad \Rightarrow \quad m_1 m_3 < m_2 m_4.$$

For all $i < j$ we have $\alpha_i < \alpha_j$, and using the general multiplicative property this implies in particular that $\sum y'_{ij} \alpha_i < \sum y'_{ij} \alpha_j$. On the other hand, $\alpha_{ij} = \alpha_j - \alpha_i$

so (A.2) can be written as $\sum y'_{ij}\alpha_j + b = \sum y'_{ij}\alpha_i$, that is, $\sum y'_{ij}\alpha_j < \sum y'_{ij}\alpha_i$. This contradiction shows that $C \neq \{0\}$.

Now choose a nonzero vector $\omega \in C$, and recursively define the following sequence of vectors:

$$\begin{aligned}\omega^0 &:= \omega, \\ \tilde{\omega}^i &:= \frac{\omega^i}{\max_{j=1,\dots,n} |\omega_j^i|}, \\ \hat{\omega}^i &:= \text{st}(\tilde{\omega}^i), \\ \omega^{i+1} &:= \tilde{\omega}^i - \hat{\omega}^i.\end{aligned}$$

Here $\text{st}(x)$ denotes the standard part of x . Every $\tilde{\omega}^i$ has at least one coefficient equal to ± 1 , and the corresponding coefficient of $\hat{\omega}^i$ is also ± 1 , so ω^{i+1} becomes zero there, and stays zero for increasing i . Hence, the number of zero entries in ω^i is at least i . Now assume that for some smallest $k < n$ we have that ω^{k+1} is the zero vector. We claim that $\eta := \sum_{i=0}^k Z^{k-i}\hat{\omega}^i$ is the required vector of polynomials. Observe that the components of η are polynomials with standard coefficients, of degree at most $n-1$.

We can write $\omega = \sum_{i=0}^k \zeta_i \hat{\omega}^i$, where the ζ_i are positive constants. Note that ζ_{i+1}/ζ_i is infinitesimal, since ω^{i+1} is a vector with infinitesimal coefficients for $i = 0, \dots, k-1$. Now let α and β be distinct standard integer vectors with $\alpha < \beta$. Because M is unbounded we have that $\alpha, \beta \in [0, M-1]^n$, and this implies that $\alpha <_\omega \beta$, or

$$\sum_{i=0}^k \zeta_i \hat{\omega}^i \cdot (\alpha - \beta) < 0.$$

Now let $0 \leq t \leq k$ be the smallest integer such that $\sum_{i=0}^t \zeta_i \hat{\omega}^i \cdot (\alpha - \beta) \neq 0$. This means that $\hat{\omega}^i \cdot (\alpha - \beta) = 0$ for $i = 0, \dots, t-1$, and $\hat{\omega}^t \cdot (\alpha - \beta) \neq 0$. Dividing by ζ_t , we can write

$$\hat{\omega}^t \cdot (\alpha - \beta) < - \sum_{i=t+1}^k \frac{\zeta_i}{\zeta_t} \hat{\omega}^i \cdot (\alpha - \beta).$$

The left hand side is standard and nonzero, the right hand side is infinitesimal, so we find $\hat{\omega}^t \cdot (\alpha - \beta) < 0$. Together with $\hat{\omega}^i \cdot (\alpha - \beta) = 0$ for $i = 0, \dots, t-1$ this implies that $\alpha <_\eta \beta$. This proves that $<_\eta$ coincides with $<$ for any pair of standard integer vectors. Since both term orders are standard, by the Transfer axiom of nonstandard analysis they are identical. ■

A.2 Proof of Proposition 5.8

We need the following version of Nakayama's lemma. For the proof see e.g. [Mar82, Ch. 1], or [Was74, p. 8].

Lemma A.4 (Nakayama). *Let K and L be \mathcal{E}^Γ -modules, then*

$$K + \mathfrak{m}L \supset L \quad \Rightarrow \quad K \supset L.$$

Here \mathfrak{m} is the unique maximal ideal in \mathcal{E}^Γ of germs of functions vanishing at the origin. We also need the following lemma. It is a symmetric version of the fundamental geometric lemma. See [Mar82] for a proof.

Lemma A.5. (Symmetric geometric lemma) *Let $F(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a t -dependent family of Γ -invariant functions, defined on a neighborhood of $(t, x) \in [0, 1] \times \{0\}$, and suppose there exists a vector field $X \in \mathbf{V}_n^\Gamma$ of the form*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^n X_i(t, x) \mathbf{v}_i$$

(where X_i are Γ -invariant families of functions and \mathbf{v}_i are generators of \mathbf{V}_n^Γ as a module over \mathcal{E}_n^Γ), defined on a neighborhood $(t, x) \in [0, 1] \times \{0\}$, such that $XF = 0$. Then there exists a Γ -equivariant germ of a diffeomorphism $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $F(0, \phi(x)) = F(1, x)$ and $\phi(0) = 0$.

Proof (of Proposition 5.8. Parts (a) and (b) are based on [Mar82, IV.4.2]): We first introduce some notation. Let l be the integer such that $M = M_k \oplus \cdots \oplus M_l$. Let π denote the projection $\pi : \mathfrak{m}_k \rightarrow M$. Let $\alpha_{im} \in \mathfrak{m}$ be homogeneous germs such that α_{im} is of degree m , and such that the set $\{\pi\alpha_{im}\}_{im}$ forms a basis of M . The generators of \mathbf{V}_n^Γ are \mathbf{v}_i , in particular $T_f = \langle \mathbf{v}_i(f) \rangle_{\mathcal{E}_n^\Gamma}$.

We write $g = f + h$, where g is the germ that is supposedly isomorphic to f . We have $h \in \mathfrak{m}_k$ by hypothesis.

(a, first part) The first part consists of proving that $T_{f+th} \supset \mathfrak{m}_k$ for $t \in [0, 1]$. By hypothesis, $T_f \supset \mathfrak{m}_k$, so we can find λ_{ijm} such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \mathbf{v}_i(f).$$

Next, define the linear operator H on M by

$$H\alpha_{im} := \pi \sum_j \lambda_{ijm} \mathbf{v}_i(h).$$

Using this we find

$$\begin{aligned} \text{(A.3)} \quad \pi T_{f+th} &\supset \text{span}_{\mathbb{R}} \left\{ \sum_j \lambda_{ijm} \mathbf{v}_i(f + th) \right\}_{im} = \\ &\text{span}_{\mathbb{R}} \{ (I + tH) \alpha_{im} \}_{im} = \text{span}_{\mathbb{R}} \{ \alpha_{im} \}_{im} = M. \end{aligned}$$

The penultimate equality holds, for $t \in [0, 1]$, if $I + tH$ is invertible for these values of t , which is true if $\pi h = \pi(g - f) \in M$ is small enough. Equation (A.3) can also be written as

$$T_{f+th} + \mathfrak{m} \cdot \mathfrak{m}_k \supset \mathfrak{m}_k,$$

and, by Nakayama, this implies $T_{f+th} \supset \mathfrak{m}_k$, proving the first part.

(a, second part) As $h \in \mathfrak{m}_k$, the statement $T_{f+th} \supset \mathfrak{m}_k$ implies that, for any $\tau \in [0, 1]$, we can find germs $X_i(t, x) \in \mathcal{E}_{1+n}^F$ defined on some neighborhood of $(t, x) = (\tau, 0)$, so that

$$\sum_i X_i \mathbf{v}_i(f + th) = -h.$$

Now write $F(t, x) = f(x) + th(x)$, and define the vector field $X := \frac{\partial}{\partial t} + \sum_i X_i(t, x) \mathbf{v}_i$, then $XF = 0$. By compactness of $[0, 1]$ we can find a finite number of such vector fields that can be combined to one defined on the entire interval. Lemma A.5 now provides the required isomorphism between $F(0, \cdot) = f$ and $F(1, \cdot) = f + h = g$.

(b) The hypothesis $\mathfrak{m} \cdot T_f \supset \mathfrak{m}_k$ implies that there exist $\lambda_{ijm} \in \mathfrak{m}$ such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \mathbf{v}_i(f).$$

As $h \in \mathfrak{m}_k$ we also have $\mathbf{v}_i(h) \in \mathfrak{m}_k$, so $\lambda_{ijk} \mathbf{v}_i(h) \in \mathfrak{m}^\Gamma \mathfrak{m}_k^\Gamma$. But $T_{f+th} = \langle \alpha_{im} + t \sum_j \lambda_{ijm} \mathbf{v}_i(h) \rangle_{\mathcal{E}_n^\Gamma}$, that is, $T_{f+th} + \mathfrak{m} \cdot \mathfrak{m}_k \supset \mathfrak{m}_k$, and by Nakayama this implies $T_{f+th} \supset \mathfrak{m}_k$. The rest of the proof is the same as the second part of (a).

(c) We assume that the \mathbf{v}_i are homogeneous. (If not, note that $\mathbf{V}^\Gamma / (\mathfrak{m} \cdot \mathbf{V}^\Gamma)$ is finite dimensional, and write $\mathbf{v}_i = \sum_j \mathbf{v}_{ij} + \mathbf{v}_{i,\text{rest}}$ where \mathbf{v}_{ij}^0 are finitely many homogeneous terms, and $\mathbf{v}_{i,\text{rest}}$ is an element of $\mathfrak{m} \mathbf{V}^\Gamma$, so that $\langle \mathbf{v}_{ij} \rangle_{\mathcal{E}_n^\Gamma} + \mathfrak{m} \cdot \mathbf{V}^\Gamma = \mathbf{V}^\Gamma$. Now use Nakayama to conclude that the \mathbf{v}_{ij} generate \mathbf{V}^Γ over \mathcal{E}_n^Γ , then use these \mathbf{v}_{ij} instead of the \mathbf{v}_i .)

Write f_k for the homogeneous k th degree part of f . We will prove the equivalence $f_k \sim_\Gamma g$. The same argument with $g = f$ then proves $f_k \sim_\Gamma f$, completing the proof.

First we prove that $T_{f_k} \supset \mathfrak{m}_k$. By hypothesis $h := f - f_k \in \mathfrak{m} \cdot \mathfrak{m}_k$, so we can write $h = h_1 h_k$ with $h_1 \in \mathfrak{m}_i$. \mathbf{v}_i maps \mathfrak{m}_j into itself, so $\mathbf{v}_i(h) = h_1 \mathbf{v}_i(h_k) + \mathbf{v}_i(h_1) h_k \in \mathfrak{m} \cdot \mathfrak{m}_k$, or $\mathbf{v}_i(f) \in T_{f_k} + \mathfrak{m} \cdot \mathfrak{m}_k$. So we have

$$\mathfrak{m}_k \subset T_f = \langle \mathbf{v}_i(f) \rangle_{\mathcal{E}_n^\Gamma} \subset T_{f_k} + \mathfrak{m} \cdot \mathfrak{m}_k.$$

Applying Nakayama we find $T_{f_k} \supset \mathfrak{m}_k$. This inclusion implies the existence of λ_{ijm} such that

$$\alpha_{im} = \sum_j \lambda_{ijm} \mathbf{v}_j(f_k),$$

and, as g_{im} , \mathbf{v}_j and f_k are homogeneous, we may assume that the λ_{ijm} are too.

Now write $g = f_k + h_k + h_{>k}$, where h_k is homogeneous of degree k , and $h_{>k}$ only contains terms of degree $k+1$ and higher. We define the operators H_k and $H_{>k}$ on M by

$$H_{(>)k} \alpha_{im} := \pi \sum_j \lambda_{ijm} \mathbf{v}_j h_{(>)k}.$$

We prove that $H_{>k}$ is nilpotent. Let $\deg(f)$ denote the total degree of a homogeneous germ f , $\text{sdeg}(f)$ the smallest total degree of terms of f , and set $\deg(0) = \text{sdeg}(0) = \infty$. Then

$$\begin{aligned} \text{sdeg}(H_{>k}\alpha_{im}) &\geq \min_j (\deg(\lambda_{ijm}) + \text{sdeg}(\mathbf{v}_j(h_{>k}))) > \\ &> \min_j (\deg(\lambda_{ijm}) + \deg(\mathbf{v}_j^0(f_k))) = \deg(\alpha_{im}) = m, \end{aligned}$$

so $H_{>k}$ maps M_m into $M_{m+1} \oplus M_{m+2} \oplus \cdots \oplus M_l$, so it is nilpotent, say $H_{>k}^{j_0} = 0$. The operator $I + t(H_k + H_{>k})$ is invertible, for $t \in [0, 1]$, if H_k is small enough, i.e., if $\pi(f_k - g) = \pi(f - g)$ is small enough. Indeed, the inverse is given by the sum

$$(A.4) \quad (I + t(H_k + H_{>k}))^{-1} = \sum_{j=0}^{\infty} (-t(H_k + H_{>k}))^j,$$

and nilpotency of $H_{>k}$ allows us to derive the inequality $\|(H_k + H_{>k})^j\| \leq C\|H_k^{j-j_0}\|$, where C is some constant, so that for small H_k , (A.4) converges. We have now:

$$\begin{aligned} \pi T_{f+t(h_k+h_{>k})} &\supset \text{span}_{\mathbb{R}}\left\{\sum_j \lambda_{ijm} \mathbf{v}_i(f + t(h_k + h_{>k}))\right\} = \\ \text{span}_{\mathbb{R}}\{(I + t(H_k + H_{>k})g_{im})\}_{im} &= \text{span}_{\mathbb{R}}\{g_{im}\} = M, \quad (t \in [0, 1]) \end{aligned}$$

where we used that $I + t(H_k + H_{>k})$ is invertible. Now apply Nakayama to conclude that $T_{f+th} \supset \mathfrak{m}_k$, where $h = h_k + h_{>k}$. The rest of the proof is the same as the second part of (a). \blacksquare